Exercise 1.4.9

Derive the integral conservation law for the entire rod with constant thermal properties by integrating the heat equation (1.2.10) (assuming no sources). Show the result is equivalent to (1.2.4).

Solution

Equation (1.2.10) in the textbook is the heat equation in one dimension without a source.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{1.2.10}$$

Replace k with $K_0/\rho c$ and multiply both sides by ρc.

$$\rho c\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2}$$

Bring ρ and c inside the time derivative.

$$\frac{\partial(\rho c u)}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2}$$

The product of mass density, specific heat, and temperature is the thermal energy density e.

$$\frac{\partial e}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2}$$

Integrate both sides over the volume V of the rod in order to get the total thermal energy in it, $\int_V e\,dV$.

$$\int_V \frac{\partial e}{\partial t}\,dV = \int_V K_0\frac{\partial^2 u}{\partial x^2}\,dV$$

Bring the time derivative in front of the volume integral. The definite volume integral wipes out the spatial variable in e, so the time derivative is a total derivative in front of the integral.

$$\frac{d}{dt}\int_V e\,dV = -\int_V \frac{\partial}{\partial x}\left(-K_0\frac{\partial u}{\partial x}\right)dV$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0\frac{\partial u}{\partial x},$$

where K_0 is a proportionality constant known as the thermal conductivity. Substitute this result into the integral on the right side.

$$\frac{d}{dt}\int_V e\,dV = -\int_V \frac{\partial \phi}{\partial x}\,dV$$

For a rod with constant cross-sectional area, the volume differential is $dV = A\,dx$.

$$\frac{d}{dt}\int_V e\,dV = -\int_a^b \frac{\partial \phi}{\partial x}A\,dx$$

Evaluate the integral on the right side.

$$\frac{d}{dt}\int_V e\,dV = A\phi(a,t) - A\phi(b,t)$$

This equation tells us that the rate the thermal energy in the rod changes is equal to the net rate that heat flows through the rod's ends.

$$\text{rate of energy accumulation} = \text{rate of energy in} - \text{rate of energy out}$$

Set $dV = A\,dx$ on the left side and then divide both sides by A to get equation (1.2.4) in the textbook.

$$\frac{d}{dt}\int_a^b e\,dx = \phi(a,t) - \phi(b,t) \tag{1.2.4}$$